The Smooth Rectangular Peg Problem

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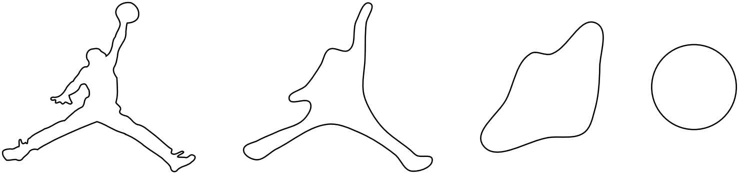
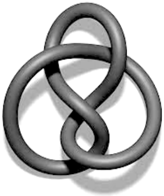
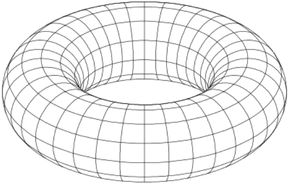
September 23, 2020

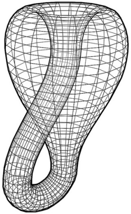
Topology concerns the properties of objects/spaces that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing.



Also concerns the way that smaller objects can sit inside of bigger ones.

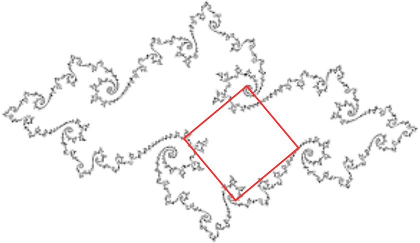


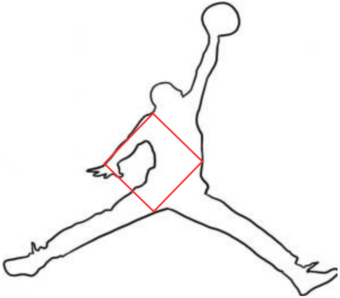




We call a continuous, closed, embedded loop in the plane a **Jordan curve**, where *embedded* means no self-intersections.

Early on, it was observed you can often find four points on a given Jordan curve that form the vertices of a square in the plane. That is, many Jordan curves have **inscribed squares**.





In 1911, Otto Toeplitz posed the following question:

The Square Peg Problem

Does every continuous Jordan curve in the plane contain four points at the vertices of a square?

Why squares?

- Three points correspond to inscribed triangles, and these are ubiquitous: Given any triangle Δ and any Jordan curve $\gamma \subset \mathbb{R}^2$, there is an inscribed triangle on γ that is similar to Δ.
- Five points correspond to inscribed pentagons, and these generally cannot be found.
- Four is more subtle. (This is a recurring theme in low-dimensional topology/geometry!)

Early progress focused on *smooth* curves, i.e., where the curve is traced out by an infinitely differentiable function $f : S^1 \to \mathbb{R}^2$. (This basically means that, if you zoom in, the Jordan curve looks like the graph of a smooth function over the x- or y-axis.)

For example:

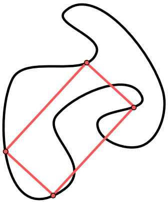
- Emch (1913) solved the problem for smooth convex curves. (Proof uses configuration spaces and homology.)
- Schnirelman (1929) solved it for any *smooth* Jordan curve.

It's tempting to try to use this to resolve the Square Peg Problem for *any* continuous Jordan curve:

Any such curve γ is the limit of *smooth* Jordan curves $\{\gamma_n\}_{n=1}^{\infty}$ that provide increasingly good approximations to γ. All these smooth curves γ_n contain squares. But these sequences of squares may shrink to points!

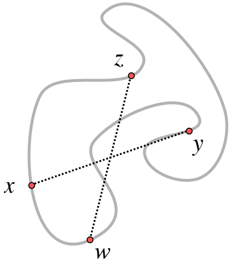
Theorem (Vaughan, 1977)

Every continuous Jordan curve contains four points forming the vertices of *some* rectangle.



The clever proof uses surfaces in 3-dimensional space.

Suppose $x, y, z, w \in \gamma$ are points forming the vertices of a rectangle R.



Observation

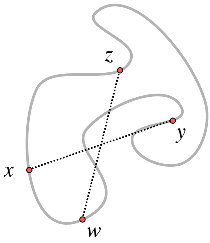
(1) the segments $x - y$ and $z - w$, which are the diagonals of R, have the same length (i.e. $\|x - y\| = \|z - w\|$), and

(2) the midpoints $(x + y)/2$ and $(z + w)/2$ are the same.

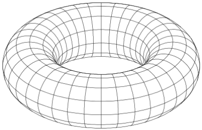
Exercise: The conditions $\|x - y\| = \|z - w\|$ and $(x + y)/2 = (z + w)/2$ are actually *equivalent* to $x, y, z, w \in \gamma$ being vertices of a rectangle.

Define a function $F : \gamma \times \gamma \to \mathbb{R}^3$ by

$$F(x, y) = \big((x + y)/2, \|x - y\|\big).$$



Note that $\gamma \times \gamma \approx S^1 \times S^1$ is a *torus*, and F sends it to $\mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$.



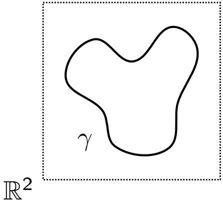
If $x, y, z, w \in \gamma$ form a rectangle as shown above, then $F(x, y) = F(z, w)$. So if there's a rectangle, this map is not injective. But actually F is never injective anyway because $F(x, y) = F(y, x)$!

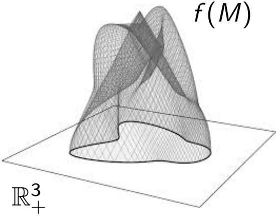
Let's fix this by considering the set M consisting of *unordered* pairs of points $\{x, y\}$ in γ.

This is a Mobius strip! The yellow boundary curve corresponds to pairs $\{x, y\}$ with $y = x$.

Since $F(x, y) = F(y, x) \implies F$ induces a map f of the Mobius strip into $\mathbb{R}^3$, i.e., $f : M \to \mathbb{R}^3_+$ defined by $f(\{x, y\}) = \left(\frac{x+y}{2}, \|x - y\|\right)$.

Example:





Observe that the boundary of M, denoted ∂M, is sent directly to γ in $\mathbb{R}^2 \times \{0\}$ because ∂M consists of unordered pairs of the form $\{x, x\}$ and

$$f(\{x,x\}) = \left(\frac{x+x}{2}, \|x-x\|\right) = (x,0).$$

In fact, $f(M) \cap (\mathbb{R}^2 \times \{0\})$ is exactly γ.

Key Claim

$f : M \to \mathbb{R}^3$ is not injective.

Non-injectivity means there are points $x, y, z, w \in \gamma$ with $\{x, y\} \neq \{z, w\}$ and $f(\{x, y\}) = f(\{z, w\})$, i.e.,

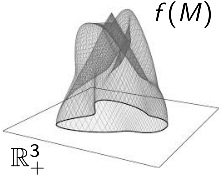
$$\frac{x+y}{2} = \frac{z+w}{2} \quad \text{and} \quad \|x-y\| = \|z-w\|.$$

These points form the vertices of a rectangle. (Note that $x, y, z, w \notin \partial M$.)

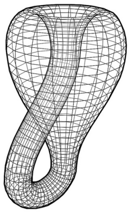
Key Claim

$f : M \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ is not injective.

Proof sketch: For the sake of contradiction, suppose that f is injective, i.e., $f(M)$ is an embedded Mobius strip in $\mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ with boundary $f(\partial M) = \gamma$ in $\mathbb{R}^2 \times 0$.



Take the mirror $-f(M) \subset \mathbb{R}^3_-$ and glue it to $f(M) \subset \mathbb{R}^3_+$ along γ. This forms a *Klein bottle*, and it is embedded in $\mathbb{R}^3$ (i.e., has no self-intersections).

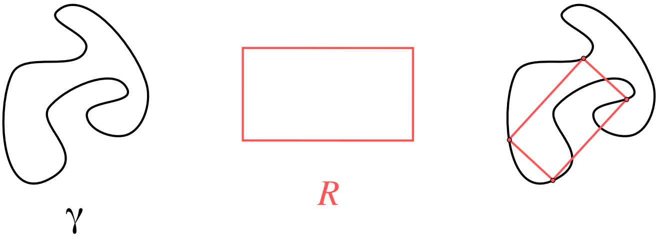


However, it is a famous old theorem in topology that the Klein bottle cannot be embedded in $\mathbb{R}^3$! $\square$

The Square Peg Problem is still open. More generally, we can ask:

The Rectangular Peg Problem (1911)

Given a Jordan curve γ and a rectangle R in the plane, does γ contain four points forming the vertices of a rectangle similar to R?



Theorem (Greene-Lobb, 2020)

Yes, if the Jordan curve is smooth!

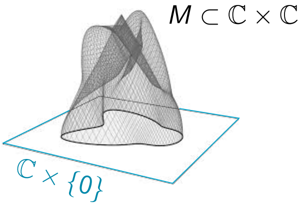
View $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ and define a map

$$F : \gamma \times \gamma \to \mathbb{C} \times \mathbb{C} \quad \text{where} \quad F(x, y) = \left(\frac{x+y}{2}, \frac{(x-y)^2}{2\sqrt{2} \cdot \|x-y\|} \right).$$

This induces a map from the Mobius strip into $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$.

For convenience, let M denote the image of the Mobius strip in $\mathbb{C} \times \mathbb{C}$.

Observe that $M \subset \mathbb{C} \times \mathbb{C}$ intersects $\mathbb{C} \times 0$ in $\partial M = \gamma \times \{0\}$.

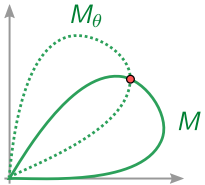


WTS: γ inscribes a rectangle whose diagonals meet at angle $\theta \in (0, \pi/2]$.

Let M_θ be obtained from M by rotating the second factor of $\mathbb{C} \times \mathbb{C}$ by θ, i.e., applying the transformation

$$\mathrm{rot}_\theta : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C} \quad \text{where} \quad (z, w) \mapsto (z, e^{i\theta} w).$$

This rotation fixes $\partial M = \gamma \subset \mathbb{C} \times \{0\}$, so M and M_θ coincide along γ.



Exercise

γ inscribes a rectangle with aspect angle θ $\iff$ M and M_θ intersect away from γ

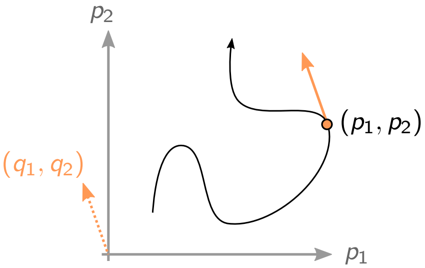
The strategy is to glue M and M_θ together along γ to form a Klein bottle.

But there's a problem: Klein bottles *do* embed in $\mathbb{R}^4$! So we have to work harder.

Fortunately, this isn't *any* old Klein bottle. But to explain why it's special, we need to discuss symplectic geometry. . .

Interlude: Symplectic geometry

Consider a point-like object moving through $\mathbb{R}^2$, tracing out some path. To understand its trajectory at any given moment, we need to know the *position* $(p_1, p_2) \in \mathbb{R}^2$ and the *momentum* (q_1, q_2), viewed as a vector "based at" (p_1, p_2) that is tangent to the path.



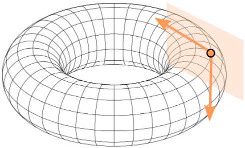
While the possibilities (p_1, p_2, q_1, q_2) form a 4-dimensional space $\mathbb{R}^4$, we no longer treat all directions equally. (In particular, our point-like object moving through $\mathbb{R}^2$ sweeps out a path in $\mathbb{R}^4$, but the physically realizable paths in $\mathbb{R}^4$ are constrained!)

In the typical geometric perspective on $\mathbb{R}^4$, the most natural way to compare two vectors $\vec{u}, \vec{v} \in \mathbb{R}^4$ is via their dot product $\vec{u} \cdot \vec{v} \in \mathbb{R}$.

In the setting of symplectic geometry, the dot product isn't the most natural way to compare directions. Instead, there is a "symplectic form" ω that eats vectors $\vec{u}, \vec{v} \in \mathbb{R}^4$ and spits out a number $\omega(\vec{u}, \vec{v}) \in \mathbb{R}$.

Example: $\omega(\vec{p}_1, \vec{q}_1) = 1$ but $\omega(\vec{p}_1, \vec{p}_1) = \omega(\vec{p}_1, \vec{p}_2) = \omega(\vec{p}_1, \vec{q}_2) = 0$

Given a surface S (like a torus or a Klein bottle) in $\mathbb{R}^4$, we can look at a point on S and consider the vectors tangent to S.



The most interesting surfaces in symplectic 4-space are **Lagrangian** surfaces, where $\omega(\vec{u}, \vec{v}) = 0$ for all vectors tangent to S at the same point.

These surfaces satisfy additional constraints.

Theorem (Shevchishin, Nemirovski 2007)

There is no smooth, embedded, Lagrangian Klein bottle in $(\mathbb{R}^4, \omega)$.

Back to our originally scheduled programming...

A direct analysis shows that the Klein bottle in $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$ constructed from γ using M and M_θ is Lagrangian.*

*Technically, it is not smooth along γ, but this can be fixed.

So if γ has no inscribed rectangles of aspect angle θ, then the Lagrangian Klein bottle $K = M \cup M_\theta$ in $\mathbb{R}^4$ has no self-intersections. That contradicts the Shevchishin-Nemirovski result. So we conclude that γ must contain an inscribed rectangle of angle of the desired angle. $\square$